Scattering and small data completeness for the critical nonlinear Schrödinger equation

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Abstract

We prove Asymptotic Completeness of one dimensional NLS with long range nonlinearities. We also prove existence and expansion of asymptotic solutions with large data at infinity.

1 Introduction

We consider the problem of scattering for the critical nonlinear Schrödinger equation in one space dimension:

$$i\partial_t v + \partial_x^2 v - \beta|v|^2 v - \gamma|v|^4 v = 0 \tag{1.1}$$

For related results in higher dimensions see e.g.[D1, G-V1, G-V2, G-V3, G-V4, GO] and cited references. In one dimension the scattering problem for NLS and/or Hartree long range type were studied before in [HKN,ST,Oz]. There are many other works in this direction, most are cited in the above references. The work closest to ours, as far as the results are concerned is [HN]. In this paper the asymptotic completeness is proved, and the L^∞ decay of the solution is shown. We use a different method, much simpler, and we get an explicit construction of the phase function and the asymptotic form of the solution as well. We also prove by the same method the existence theory of wave operators for large data in the repulsive case, and small data in the general nonlinear case. In the work of [G-V4], the Hartree equation in 3 or more dimensions is considered; the analysis uses, among other things the representation of the solution in hyperbolic coordinates which we also use. But the approach used in this paper is very different and much more involved than the work presented here. See also [Nak].

Recall first that a solution of the linear Schrödinger, i.e. $\beta = \gamma = 0$, with fast decaying smooth initial data satisfies

$$u(t,x) \sim t^{-1/2} e^{ix^2/4t}\, \widehat{u}(0, x/t) \tag{1.2}$$

where $\widehat{u}(t,\xi) = \int u(t,x)e^{-ix\xi}\, dx$ denotes the Fourier transform with respect to x only.

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We make the following ansatz for the solution of the nonlinear problem

$$v(t,x) = t^{-1/2} e^{ix^2/4t}\, V(t,y), \qquad s=t, \quad y=x/t \tag{1.3}$$

Plugging this into (1.1) gives, since $(i\partial_t + \partial_x^2)\Big(t^{-1/2}e^{ix^2/4t}\Big) = 0$,

$$\begin{aligned}(i\partial_t + \partial_x^2)v(t,x) = (i\partial_t + \partial_x^2)\Big(t^{-1/2}e^{ix^2/4t}\, V(s,y)\Big) &= t^{-1/2}e^{ix^2/4t}\big(i\partial_t + \partial_x^2 + i(x/t)\partial_x)\big)V(s,y) \\ &= t^{-1/2}e^{ix^2/4t}\big(i\partial_s + s^{-2}\partial_y^2\big)V(s,y)\end{aligned} \tag{1.4}$$

Hence (1.1) becomes

$$\Psi(V) = i\partial_s V - \beta s^{-1}|V|^2 V - \gamma s^{-2}|V|^4 V + s^{-2}\partial_y^2 V = 0 \tag{1.5}$$

It is easy to check that the general solution to the ODE

$$L(g) = i\,\frac{d}{ds}g - \frac{\beta}{s}|g|^2\, g - \frac{\gamma}{s^2}|g|^4\, g = 0$$

is of the form

$$g = ae^{i\phi}, \quad \text{where} \quad \phi = -\beta a^2 \ln|s| \, + \gamma\frac{a^4}{s} + b$$

for some constants a and b.

It is therefore natural with the following ansatz for the solution of the nonlinear problem

$$V(s,y) \sim V_0(s,y) = a(y)e^{i\phi(s,y)}, \qquad \phi(s,y) = -\beta a(y)^2 \ln|s| + b(y) \tag{1.6}$$

where $a(y)$ and $b(y)$ are any smooth sufficiently fast decaying functions of $y = x/t$.

First we show scattering, i.e. given any $a(y)$ and $b(y)$ as above we show that there is a solution V as above.

Theorem 1.1. *Suppose that $a(y)$ and $b(y)$ are polynomially decaying smooth real valued functions and let $v_0(t,x) = t^{-1/2}e^{ix^2/4t}V_0(t,x/t)$, where V_0 is given by (1.6). Then if $\beta \geq 0$ or β is small (1.1) has a smooth solution $v \sim v_0$ as $t \to \infty$, satisfying*

$$\|(v-v_0)(t,\cdot)\|_{L^\infty} + \|(v-v_0)(t,\cdot)\|_{L^2} \leq C(1+\ln(1+t))^2(1+t)^{-1} \tag{1.7}$$

We then show asymptotic completeness for small initial data, i.e. that there is a an asymptotic expansion of the form (1.6).

Theorem 1.2. *Suppose that $f \in C_0^\infty$. Then if $\varepsilon > 0$ is sufficiently small (1.1) has a global solution with data $v(0,x) = \varepsilon f(x)$. Moreover there are functions $a(y)$ and $b(y)$ such that with $v_0(t,x) = t^{-1/2}e^{ix^2/4t}V_0(t,x/t)$, where V_0 is given by (1.6), $v \sim v_0$ as $t \to \infty$;*

$$\|(v-v_0)(t,\cdot)\|_{L^\infty} \leq C(1+t)^{-3/2+C\varepsilon} \tag{1.8}$$

2 The first order asymptotics and small data existence at infinity

The ansatz we use is an approximate solution of the form

$$v_0(t,x) = s^{-1/2} e^{ix^2/4t} V_0(s,y), \quad \text{where} \quad V_0(s,y) = a(y)e^{i\phi(s,y)}, \qquad \phi(s,y) = -\beta a(y)^2 \ln|s| + b(y)$$

where $a(y)$ and $b(y)$ are any smooth sufficiently fast decaying functions of $y = x/t$ and $s = t$.

$$(2.1) \quad \big(i\,\partial_t + \partial_x^2 - \beta|v_0|^2 - \gamma|v_0|^4\big)v_0 = t^{-1/2} e^{ix^2/4t}\Big(i\,\partial_s + s^{-2}\partial_y^2 - \beta s^{-1}|V_0|^2 - \gamma s^{-2}|V_0|^4\Big)V_0$$
$$= s^{-5/2} e^{ix^2/4t}\big(-\gamma|V_0|^4 V_0 + \partial_y^2 V_0\big) = F_0.$$

Assuming that $a(y)$ and $b(y)$ decay polynomially we have

$$|\partial_s^i \partial_y^j v_0| \leq \frac{C_N(1+\beta\ln|s|)^j}{\big(s(1+|y|)\big)^{1/2}(1+|y|)^N}$$

and

$$(2.2) \qquad |\partial_s^i \partial_y^j F_0| \leq \frac{C_N\,(1+\beta\ln|s|)^{2+j}}{\big(s(1+|y|)\big)^{5/2}(1+|y|)^N}$$

for any N. It follows that

$$|\partial_t^i \partial_x^j v_0| \leq \frac{C_N}{\big(t+|x|\big)^{1/2}(1+|x/t|)^N}$$

and

$$(2.3) \qquad |\partial_t^i \partial_x^j F_0| \leq \frac{C_N\,(1+\beta\ln|t|)^2}{\big(t+|x|\big)^{5/2}(1+|x/t|)^N}$$

for any N.

We now consider

$$w = v - v_0$$

$$(i\,\partial_t + \partial_x^2)w = G(v_0,w) + F_0$$

where

$$G(v_0,w) = \beta\big(|v_0+w|^2(v_0+w) - |v_0|^2 v_0\big) + \gamma\big(|v_0+w|^4(v_0+w) - |v_0|^4 v_0\big)$$

The solution of the PDE

$$(2.4) \qquad (i\,\partial_t + \partial_x^2)w = F$$

with vanishing final data at infinity is given by

$$w(t,x) = \int_t^\infty \int E(t-s,x-y)F(s,y)\,dyds$$

where E is the forward fundamental solution of $i\partial_t + \partial_x^2$.

Lemma 2.1. *Suppose that*

$$i\partial_t w + \partial_x^2 w = F \tag{2.5}$$

Then

$$\|w(t,\cdot)\|_{L^2} \le \|w(t_0,\cdot)\|_{L^2} + \Big| \int_{t_0}^{t} \|F(s,\cdot)\|_{L^2}\, ds \Big| \tag{2.6}$$

Proof. The energy identity for this equation is

$$\frac{d}{dt}\int |w(t,x)|^2 dx = 2\int \Im(F\overline{w})(t,x)\, dx \tag{2.7}$$

where $\Im$ is the imaginary part. □

The energy estimate is therefore

$$\|w(t,\cdot)\|_{L^2} \le \int_t^\infty \|F(s,\cdot)\|_{L^2}\, ds \tag{2.8}$$

From differentiating the equation it also follows that

$$\sum_{|\alpha|\le 1} \|\partial^\alpha w(t,\cdot)\|_{L^2} \le \int_t^\infty \sum_{|\alpha|\le 1} \|\partial^\alpha F(s,\cdot)\|_{L^2}\, ds \tag{2.9}$$

We will now use the above inhomogeneous estimate together with an iterative procedure to get existence for the equation in the previous section, of a solution w decaying at infinity to zero fast, in a sense having vanishing data at infinity. We therefore put up an iteration

$$(i\partial_t + \partial_x^2)w_0 = F_0, \qquad (i\partial_t + \partial_x^2)w_{k+1} = G(v_0, w_k) + F_0$$

where the solutions are defined as convolution with the fundamental solution that vanishes at infinity (more precise later on). We must now first find the right estimates for w_0 and thereafter make an assumption that the other iterates have similar bounds. It follows from (2.3) that

$$\sum_{|\alpha|\le 1} \|\partial^\alpha F_0(t,x)\|_{L^2} \le \frac{C(1+\beta\ln|1+t|)^2}{t^2}$$

and hence

$$\int_t^\infty \sum_{|\alpha|\le 1} \|\partial^\alpha F_0(t,\cdot)\|_{L^2} dt \le \frac{K(1+\beta\ln|1+t|)^2}{t}$$

for some fixed constant K. We therefore make the inductive assumption that

$$\sum_{|\alpha|\le 1} \|\partial^\alpha w_k(t,\cdot)\|_{L^2} \le \frac{2K(1+\beta\ln|1+t|)^2}{t} \tag{2.10}$$

Lemma 2.2.

$$\|w(t,\cdot)\|_{L^\infty}^2 \le \|w(t,\cdot)\|_{L^2}\|\partial_x w(t,\cdot)\|_{L^2} \tag{2.11}$$

Proof. Follows by Hölder's inequality $w^2 \le 2\int |w||w_x|\,dx \le 2\|w\|_{L^2}\|\partial w\|_{L^2}$. □

It follows that

$$\|w_k(t,\cdot)\|_{L^\infty} \le \frac{2K(1+\beta\ln|1+t|)^2}{t}$$

Also using the estimates for v_0,

$$\sum_{|\alpha|\le 1} |\partial^\alpha v_0| \le C_0/t^{1/2}$$

we get

$$\sum_{|\alpha|\le 1} \|\partial^\alpha G(v_0,w_k)\|_{L^2} \le \frac{C_1\beta}{t}\sum_{|\alpha|\le 1}\|\partial^\alpha w_k\|_{L^2}, \qquad \text{if} \qquad t\ge t_0$$

for some number $t_0 = t_0(\beta) < \infty$. t_0 is chosen such that the r.h.s. of equation (2.10) is smaller than 1. Hence by the energy inequality and the inductive assumption we get for $t \ge t_0'(\beta)$;

$$\begin{aligned}\sum_{|\alpha|\le 1}\|\partial^\alpha w_{k+1}(t,\cdot)\|_{L^2} &\le \int_t^\infty \frac{C_1\beta}{s}\sum_{|\alpha|\le 1}\|\partial^\alpha w_k(s,\cdot)\|_{L^2}\,ds + \frac{K(1+\ln|1+t|)^2}{t},\\ &\le \int_t^\infty \frac{C_1\beta 2K(1+\beta\ln|1+s|)^2ds}{s^2} + \frac{K(1+\ln|1+t|)^2}{t} \le \frac{(C_2\beta+1)K(1+\ln|1+t|)^2}{t}\end{aligned} \tag{2.12}$$

Hence (2.10) follows also for k replaced by $k+1$, if β is so small that $C_2\beta\le 1$. This proves the theorem for small β.

3 Global existence and decay for the initial value problem

Here we show that (1.1) has a global solution for small initial data and that the solution decays like $t^{-1/2}$. Let us suppose we are given initial data when $s=t=1$,say in C_0^∞ .

Lemma 3.1. *Suppose that g is real valued and*

$$i\partial_s V - gV = F. \tag{3.1}$$

Then

$$|V(s)| \le |V(s_0)| + \Big|\int_{s_0}^s |F(\sigma)|\,d\sigma\Big|. \tag{3.2}$$

Proof. Multiplying with the integrating factor $e^{iG(s)}$, where $G=\int g\,ds$ gives $\partial_s\big(Ve^{iG}\big) = -i\,F\,e^{iG}$ and the lemma follows from integrating this. □

It now follows that if (1.5) holds then

$$|V(s,y)| \leq |V(1,y)| + \int_1^s |\partial_y^2 V(\sigma,y)| \frac{d\sigma}{\sigma^2}. \tag{3.3}$$

Hence the desired bound for V would follow if we can prove that for some fixed $\delta < 1$;

$$|\partial_y^2 V(s,y)| \leq C\varepsilon(1+s)^\delta. \tag{3.4}$$

We will now derive this bound from energy bounds. We will assume that

$$|V| \leq C_0\varepsilon \tag{3.5}$$

Writing (1.5) in the form

$$i\partial_s V - s^{-2}\partial_y^2 V = F = \beta s^{-1}|V|^2\, V + \gamma s^{-2}|V|^4 V \tag{3.6}$$

and differentiating the above equation with respect to y gives

$$\big(i\partial_s - s^{-2}\partial_y^2\big)V^{(k)} = F^{(k)}, \qquad V^{(k)} = \partial_y^k V, \quad F^{(k)} = \partial_y^k F \tag{3.7}$$

We claim that

$$\|F^{(k)}(s,\cdot)\|_{L^2} \leq Cs^{-1}\big(1 + s^{-1}\|V(s,\cdot)\|_{L^\infty}^2\big)\|V(s,\cdot)\|_{L^\infty}^2 \|V^{(k)}(s,\cdot)\|_{L^2}, \quad k = 0,1,2,3. \tag{3.8}$$

In fact

$$|F^{(0)}| \leq Cs^{-1}(1+s^{-1}|V|^2)|V|^3 \tag{3.9}$$

$$|F^{(1)}| \leq Cs^{-1}(1+s^{-1}|V|^2)|V|^2|\partial_y V| \tag{3.10}$$

$$|F^{(2)}| \leq Cs^{-1}(1+s^{-1}|V|^2)\big(|V|^2\,|\partial_y^2 V| + |V|\,|\partial_y V|^2\big) \tag{3.11}$$

$$|F^{(3)}| \leq Cs^{-1}(1+s^{-1}|V|^2)\big(|V|^2|\partial_y^3 V| + |V|\,|\partial_y V|\,|\partial_y^2 V| + |\partial_y V|^3\big) \tag{3.12}$$

For $k = 0,1$ this is obvious and for $k \geq 2$ this follows from interpolation (proved by just integrating by parts):

Lemma 3.2.

$$\|\partial_y^j V\|_{L^{2k/j}}^{k/j} \leq C\|V\|_{L^\infty}^{k/j-1}\|\partial_y^k V\|_{L^2} \tag{3.13}$$

For $k = 2$ we have

$$\|\partial_y V(t,\cdot)\|_{L^4}^2 \leq C\|V(t,\cdot)\|_{L^\infty}\|\partial_y^2 V(t,\cdot)\|_{L^2}, \tag{3.14}$$

Similarly for $k = 3$ we have

$$\|\partial_y V(t,\cdot)\|_{L^6}^3 \leq C\|V(t,\cdot)\|_{L^\infty}^2\|\partial_y^3 V(t,\cdot)\|_{L^2}, \tag{3.15}$$

$$\|\partial_y^2 V(t,\cdot)\|_{L^3}^{3/2} \leq C\|V(t,\cdot)\|_{L^\infty}^2\|\partial_y^3 V(t,\cdot)\|_{L^2}. \tag{3.16}$$

Lemma 3.3. *Suppose that*

$$i\partial_s W - s^{-2}\partial_y^2 W = F. \tag{3.17}$$

Then

$$\|W(s,\cdot)\|_{L^2} \le \|W(s_0,\cdot)\|_{L^2} + \Big|\int_{s_0}^{s} \|F(\sigma,\cdot)\|_{L^2}\, d\sigma\Big|. \tag{3.18}$$

Assuming the bound

$$\|V(s,\cdot)\|_{L^\infty} \le K\varepsilon = \delta \tag{3.19}$$

with a constant independent of s we have hence proven, using the above lemma and equations (3.7)-(3.16) that

$$\|V^{(k)}(t,\cdot)\|_{L^2} \le \|V^{(k)}(1,\cdot)\|_{L^2} + \int_1^t D_k(\delta+\delta^2)\|V^{(k)}(\tau,\cdot)\|_{L^2}\,\tau^{-1}d\tau, \tag{3.20}$$

from which it follows that

$$\|V^{(k)}(t,\cdot)\|_{L^2} \le C_k\varepsilon(1+t)^{D_k(\delta+\delta^2)}, \qquad k=0,1,2,3. \tag{3.21}$$

and by Lemma 2.2

$$\|V^{(k)}(t,\cdot)\|_{L^\infty} \le C_{k+1}\varepsilon(1+t)^{D_k(\delta+\delta^2)}, \qquad k=0,1,2 \tag{3.22}$$

where C_k is a constant such that $\sum_{j\le k}\|V^{(j)}(1,\cdot)\|_{L^2} \le C_k\varepsilon$. Now suppose that $\varepsilon>0$ is so small that

$$D_3\big(4C_3\varepsilon + (4C_3\varepsilon)^2\big) \le 1/4. \tag{3.23}$$

It then follows from (3.3) that

$$\|V(s,\cdot)\|_{L^\infty} \le 4C_3\varepsilon \qquad \text{and} \qquad \|\partial_y^2 V(s,\cdot)\|_{L^\infty} \le C_3\varepsilon(1+t)^{1/2}. \tag{3.24}$$

This is more than needed in (3.4).

4 The completeness

Lemma 4.1. *Suppose that g is real valued and*

$$i\partial_s V - gV = F \tag{4.1}$$

Then with $G(s,y)=\int_{s_0}^s g(\tau,y)\,d\tau$

$$\big|\partial_s|V|\big| + \big|\partial_s(Ve^{iG})\big| \le 2|F| \tag{4.2}$$

Proof. Multiplying with $\overline{V}$ gives

$$i\partial_s|V|^2 = \Im F\overline{V} \tag{4.3}$$

and it follows that $|\partial_s|V|| \le |F|$. Multiplying with the integrating factor $e^{iG(s)}$, where $G=\int g\,ds$ gives $\partial_s\big(Ve^{iG}\big) = -i\,F\,e^{iG}$ and the lemma follows. $\square$

In the application $g = -\beta s^{-1}|V|^2 - \gamma s^{-2}|V|^4$ and $F = s^{-2}\partial_y^2 V$. We already have proven that

$$|F(s,y)| \le C\varepsilon s^{-2+C\varepsilon^2} \tag{4.4}$$

in the previous section. It therefore follows from the above lemma that the limit exists

$$\Big||V(s,y)| - a(y)\Big| \le C\varepsilon s^{-1+C\varepsilon^2}, \qquad \text{where} \quad a(y) = \lim_{s\to\infty} |V(s,y)| \tag{4.5}$$

It therefore also follows from the lemma that

$$|G(s,y) - \phi(s,y)| \le C\varepsilon s^{-1+C\varepsilon^2}, \qquad \text{where} \quad \phi(s,y) = a(y)^2\beta \ln|s| + b(y) \tag{4.6}$$

and $-b(y)$ is defined as the limit of $G(s,y) - a(y)^2\beta\ln|s|$ as $s\to\infty$. Hence

$$\Big|V(s,y) - a(y)e^{i\phi(s,y)}\Big| \le C\varepsilon s^{-1+C\varepsilon^2} \tag{4.7}$$

5 Higher order asymptotics and large data existence at infinity

We now want to construct a higher order asymptotic expansion at infinity. Therefore, we want to linearize the operator

$$L(g) = i\frac{d}{ds}g - \frac{\beta}{s}G_1(g) - \frac{\gamma}{s^2}G_2(g), \qquad G_1(g) = |g|^2 g, \qquad G_2(g) = |g|^4 g$$

We have $G_i(V_0+W) = G_i(V_0) + G_i'(V_0)W + O(|W|^2)$, where

$$G_1'(V_0)W = 2|V_0|^2 W + V_0^2\overline{W}, \qquad G_2'(V_0)W = 3|V_0|^4 W + 2|V_0|^2 V_0^2\overline{W}$$

Here,

$$V_0 = a(y)e^{i\phi(s,y)} \qquad \phi(s,y) = -\beta a(y)^2 \ln|s| + b(y)$$

Hence the linearized operator is

$$L_0 W = L'(V_0)W = i\frac{d}{ds}W - \frac{\beta}{s}G_1'(V_0)W - \frac{\gamma}{s^2}G_2'(V_0)W$$

Observe that L_0 is not complex linear. If Z is constant it therefore follows that $(k\ge 1)$

$$L_0\Big(\frac{e^{i\phi}\ln^j|s|}{s^k}Z\Big) = \frac{e^{i\phi}\ln^j|s|}{s^{k+1}}\big((-2\beta a^2 - ik)Z - \beta a^2\overline{Z}\big) + i\,j\frac{e^{i\phi}\ln^{j-1}|s|}{s^{k+1}}Z + \frac{e^{i\phi}\ln^j|s|}{s^{k+2}}\big(-3\gamma a^2 Z - 2\gamma a^4\overline{Z}\big)$$

The inverse of

$$(-2\beta a^2 - ik)Z - \beta a^2\overline{Z} = Y$$

is given by

$$Z = \frac{1}{k^2+3\beta^2a^4}\big(-2\beta a^2 + i\,k\big)Y + \frac{1}{k^2+3\beta^2a^4}\beta a^2\overline{Y}$$

and hence

(5.1)
$$L_0\Big(\frac{e^{i\phi}\ln^j|s|}{s^k(k^2+3\beta^2a^4)}\Big(\big(-2\beta a^2+i\,k\big)Y+\beta a^2\overline{Y}\Big)\Big)=\frac{e^{i\phi}\ln^j|s|}{s^{k+1}}Y+i\,j\frac{e^{i\phi}\ln^{j-1}|s|}{s^{k+1}(k^2+3\beta^2a^4)}\Big(\big(-2\beta a^2+i\,k\big)Y+\beta a^2\overline{Y}\Big)$$
$$+\frac{e^{i\phi}\ln^j|s|}{s^{k+2}(k^2+3\beta^2a^4)}\gamma a^2\big((2\beta a^2-i\,k)[3Y+2a^2\overline{Y}]+\beta a^2(2a^2Y-3\overline{Y})\big)$$

It follows that

Lemma 5.1. *Let $\mathcal{S}_k$ denote a finite sum of the form ($k\geq 1$)*

$$\sum_{k'\geq k,\,j\geq 0} c_{k'j}(y)\frac{e^{i\phi}\,\ln^j|s|}{s^{k'}},\qquad \phi=-\beta a(y)^2\ln|s|+b(y), \tag{5.2}$$

with coefficients decaying polynomially in y. More precisely $|\partial^\alpha c_{jk'}(y)|\leq C_N(1+|y|)^{-N}$, for any N and $c_{jk'}=0$ for k', j sufficiently large. Here $\ln^0|s|=1$.

Then if $k\geq 1$ and $\psi_{k+1}\in\mathcal{S}_{k+1}$ there is $\phi_k\in\mathcal{S}_k$ and $\psi_{k+2}\in\mathcal{S}_{k+2}$ such that

$$L_0\,\phi_k=\psi_{k+1}+\psi_{k+2} \tag{5.3}$$

Recall that $\Psi(V)=L(V)+s^{-2}\partial_y^2V$ and that $L_0=L'(V_0)$. We have

Lemma 5.2. *Let $\Psi_n=\Psi'(V_n)$ and suppose that $V_n-V_0\in\mathcal{S}_1$. Then if $k\geq 1$ and $\psi_{k+1}\in\mathcal{S}_{k+1}$ there is $\phi_k\in\mathcal{S}_k$ and $\psi_{k+2}\in\mathcal{S}_{k+2}$ such that*

$$\Psi_n\,\phi_k=\psi_{k+1}+\psi_{k+2} \tag{5.4}$$

Proof. First, let $\phi_k\in\mathcal{S}_k$ be as in the previous lemma. Then $(\Psi_0-L_0)\phi_k=s^{-2}\partial_y^2\phi_k\in\mathcal{S}_{k+2}$. Furthermore $\Psi_n-\Psi_0=s^{-1}\beta G'(V_n)-s^{-1}\beta G'(V_0)=s^{-1}O(V_n-V_0)\in\mathcal{S}_2$ so $\big(\Psi_n-\Psi_0\big)\phi_k\in\mathcal{S}_{k+2}$. $\square$

By the results of previous sections, $\Psi(V_0)\in\mathcal{S}_2$. See e.g. equation (2.1). We will now inductively, for $n\geq 1$ construct V_n such that $V_n-V_0\in\mathcal{S}_1$ and $\Psi(V_n)\in\mathcal{S}_{n+2}$. Assume that this is true for $n\leq k$. Then by the above lemma (with $\psi_{k+1}=\Psi(V_k)\in\mathcal{S}_{k+2}$) we can find V_{k+1} such that

$$\Psi(V_k)+\Psi'(V_k)(V_{k+1}-V_k)\in\mathcal{S}_{k+3},\qquad V_{k+1}-V_k\in\mathcal{S}_{k+1}. \tag{5.5}$$

Furthermore, there are bilinear forms in (X,Z); $G_i''(U,V)(X,Z)$ such that

$$G_i(U)=G_i(V)+G_i'(V)(U-V)+G_i''(U,V)(U-V,U-V) \tag{5.6}$$

Then

$$\Psi(U)=\Psi(V)+\Psi'(V)(U-V)-\frac{\beta}{s}G_1''(U,V)(U-V,U-V)-\frac{\gamma}{s^2}G_2''(U,V)(U-V,U-V). \tag{5.7}$$

Hence

(5.8)
$$\Psi(V_{k+1})=\Psi(V_k)+\Psi'(V_k)(V_{k+1}-V_k)$$
$$-\frac{\beta}{s}G_1''(V_{k+1},V_k)(V_{k+1}-V_k,V_{k+1}-V_k)-\frac{\gamma}{s^2}G_2''(V_{k+1},V_k)(V_{k+1}-V_k,V_{k+1}-V_k)\in\mathcal{S}_{k+3}.$$

Let

$$v_k(t,x) = t^{-1/2} e^{ix^2/4t}\, V_k(t,y), \qquad s=t, \quad y = x/t \tag{5.9}$$

Then (see equation (2.1))

$$i\partial_t v_k + \partial_x^2 v_k - \beta |v_k|^2 v_k - \gamma |v_k|^4 v_k = t^{-1/2} e^{ix^2/4t} \Psi(V_k) = F_k \tag{5.10}$$

It follows that

$$|\partial^\alpha F_k| \le \frac{C_k}{(t+|x|)^{2+k}} \tag{5.11}$$

and hence

$$\sum_{|\alpha|\le 1} \|\partial^\alpha F_N(t,\cdot)\|_{L^2} \le \frac{K_N}{t^N}$$

for some constant K_N. We then define $w_0 = 0$ and for $k \ge 1$:

$$(i\partial_t + \partial_x^2) w_{k+1} = \beta G(v_N, w_k) w_k + F_N, \qquad k \ge 0. \tag{5.12}$$

We will inductively assume that

$$\|\partial w_k(t,\cdot)\|_{L^2} + \|w_k(t,\cdot)\|_{L^2} \le \frac{4K_N}{N t^N} \tag{5.13}$$

Since by Hölder's inequality

$$w^2 \le 2\int |w||w_x|\, dx \le 2\|w\|_{L^2}\|\partial w\|_{L^2} \le \|\partial w\|_{L^2}^2 + \|w\|_{L^2}^2$$

we also get

$$\|w_k(t,\cdot)\|_{L^\infty} \le \frac{4K_N}{N\, t^N}$$

Since also

$$|V_0| = |a(y)| \le C_0$$

it follows that

$$\|v_N(t,\cdot)\|_{L^\infty} = t^{-1/2}\|V_N(t,\cdot)\| \le t^{-1/2}[\|V_0\| + C_N t^{-1}] \le \frac{2C_0}{t^{1/2}}, \qquad t \ge t_N = \frac{C_N}{C_0}$$

since by construction, $V_N - V_0 \in \mathcal{S}_1$. So C_0 is independent of N, if t_N is sufficiently large. It follows that

$$\|G(v_1, w_k)(t,\cdot)\|_{L^\infty} \le \frac{8C_0}{t}, \qquad t \ge t_N' \tag{5.14}$$

Hence by the energy inequality (2.9), (5.12-14)

$$\|\partial w_{k+1}(t,\cdot)\|_{L^2} + \|w_{k+1}(t,\cdot)\|_{L^2} \le \int_t^\infty \frac{\beta 8 C_0}{s}\frac{4K_N}{N s^N} ds + \frac{K_N}{s^{N+1}}\, ds = \Big(\frac{32\beta\, C_0}{N} + 1\Big)\frac{K_N}{N\, t^N} \le \frac{2K_N}{N t^N}, \qquad t \ge t_N''$$

if $\beta > 0$ is sufficiently small and t_N'' is sufficiently large. Hence (5.13) follows also for $k+1$.

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